MATEMATIKA EKONOMI

FUNGSI MULTIVARIABEL DAN PENGOPTIMUMANNYA

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Turunan Parsial

- Misalkan diberikan fungsi 2-variabel:

$$z = f(x, y).$$

- Turunan parsial z terhadap x dan y berturut-turut ialah:

$$f_1 = f_x = \frac{\partial z}{\partial x}, \quad f_2 = f_y = \frac{\partial z}{\partial y}.$$

- Dalam penurunan parsial, jika variabel x berubah maka variabel y dianggap tetap (konstan)
- Contoh:

$$f(x, y) = x^2 y \Leftrightarrow f_x = 2xy, \quad f_y = x^2.$$

Turunan Parsial

- Tentukan dan jelaskan turunan-turunan parsial dari fungsi produksi berikut:

$$Y = 10X_1^{1/2}X_2^{1/2}.$$

- Diperoleh:

$$\frac{\partial Y}{\partial X_1} = 5X_1^{-1/2}X_2^{1/2} = \frac{5X_2^{1/2}}{X_1^{1/2}} = 5\left[\frac{X_2}{X_1}\right]^{1/2}, \quad \frac{\partial Y}{\partial X_2} = 5\left[\frac{X_1}{X_2}\right]^{1/2}.$$

- Dari turunan parsial yang pertama: semakin besar X_2 semakin besar tambahan output yang diakibatkan oleh kenaikan penggunaan X_1.

$$\left.\frac{\partial Y}{\partial X_1}\right|_{(25,9)} > \left.\frac{\partial Y}{\partial X_1}\right|_{(25,4)}.$$

Turunan Parsial

- Fungsi produksi CES:

$$Y = A[bX_1^{\theta} + (1-b)X_2^{\theta}]^{1/\theta}, \quad A > 0, 0 < b < 1, \theta < 1.$$

- Diperoleh:

$$\frac{\partial Y}{\partial X_1} = bA^{\theta}\left[\frac{Y}{X_1}\right]^{1-\theta},$$

$$\frac{\partial Y}{\partial X_2} = (1-b)A^{\theta}\left[\frac{Y}{X_2}\right]^{1-\theta}.$$

Turunan Parsial Kedua

- Misalkan diberikan fungsi 2-variabel:

$$z = f(x, y).$$

- Turunan parsial kedua z terhadap x dan y berturut-turut ialah:

$$f_{11} = f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial f_x}{\partial x}, \quad f_{22} = f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial f_y}{\partial y},$$

$$f_{12} = f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial f_x}{\partial y}, \quad f_{21} = f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f_y}{\partial x}.$$

Turunan Parsial Kedua

- Tentukan turunan-turunan parsial pertama dan kedua dari fungsi berikut

$$f(x, y) = x^2 y.$$

- Turunan parsial pertama:

$$f_1 = 2xy, \quad f_2 = x^2.$$

- Turunan parsial kedua:

$$f_{11} = 2y, \quad f_{12} = 2x,$$
$$f_{21} = 2x, \quad f_{22} = 0.$$

Teorema Young: $f_{ij} = f_{ji}$.

Vektor gradien:

$$\nabla f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 2xy \\ x^2 \end{bmatrix}.$$

Matriks gradien:

$$\nabla_2 f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} 2y & 2x \\ 2x & 0 \end{bmatrix}.$$

Turunan Total

- Diberikan fungsi:

$$z = f(x, y).$$

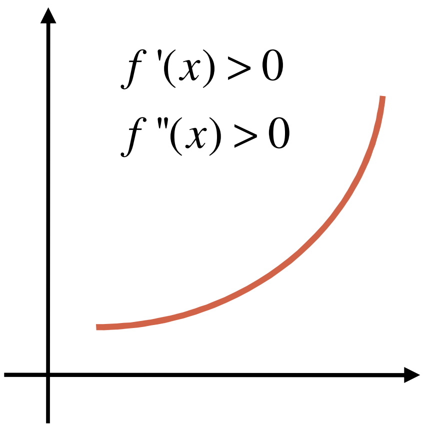
- Turunan total pertama

$$dz = f_x dx + f_y dy.$$

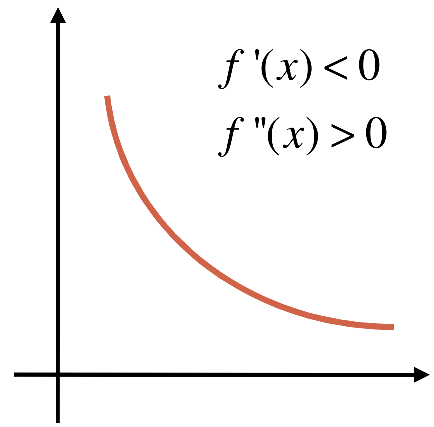
- Turunan total kedua

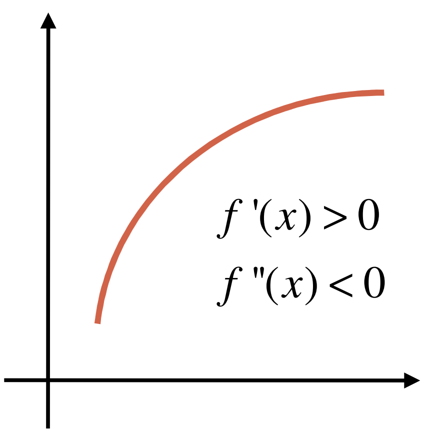
$$\begin{aligned} d^2 z &= d(f_x dx + f_y dy) \\ &= f_{xx} dx^2 + f_{yx} dydx + f_{xy} dxdy + f_{yy} dy^2 \\ &= f_{xx} dx^2 + 2 f_{xy} dxdy + f_{yy} dy^2. \end{aligned}$$

Kecekungan

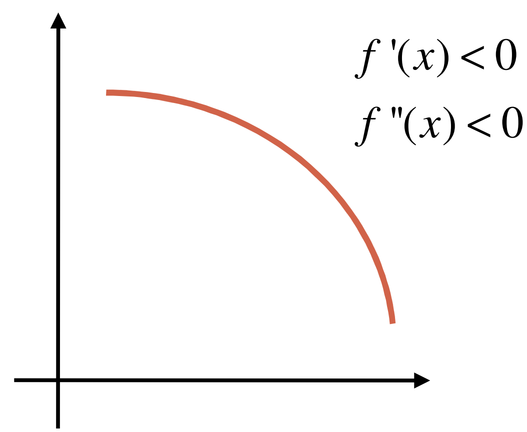


Cekung ke atas (konveks)





Cekung ke bawah (konkaf)



Kecekungan

- Diberikan fungsi:

$$z = f(x, y).$$

- Fungsi f di atas cekung ke atas (konveks) jika dan hanya jika

$$d^2 z = f_{xx} dx^2 + 2 f_{xy} dxdy + f_{yy} dy^2 \geq 0.$$

- Fungsi f di atas cekung ke bawah (konkaf) jika dan hanya jika

$$d^2 z = f_{xx} dx^2 + 2 f_{xy} dxdy + f_{yy} dy^2 \leq 0.$$

- Contoh: Diberikan fungsi:

$$f(x, y) = 5 - (x + y)^2.$$

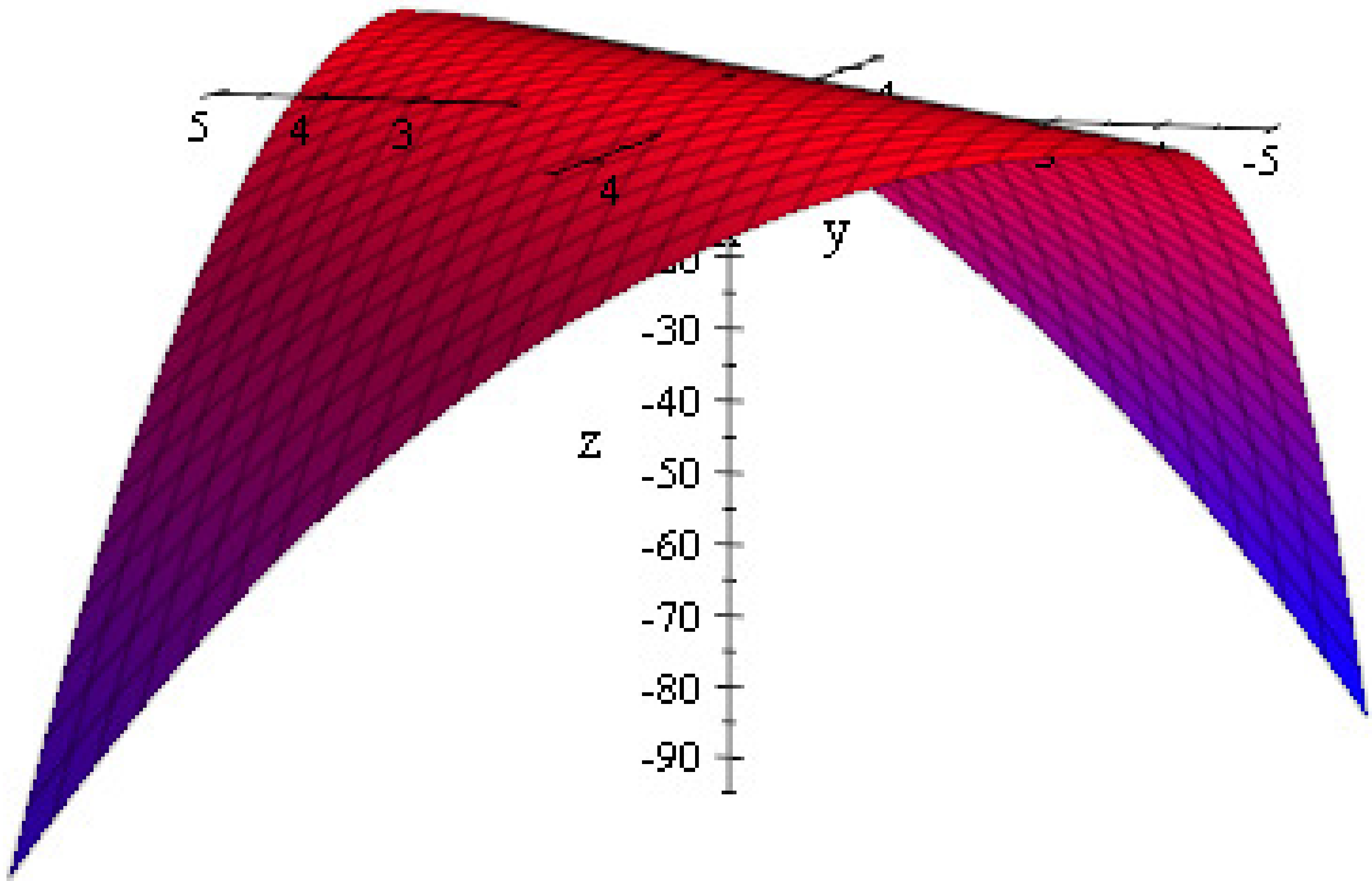
Diperoleh:

$$f_1 = f_2 = -2(x + y),$$

$$f_{11} = f_{22} = f_{12} = -2,$$

$$d^2 z = -2(dx + dy)^2 \leq 0.$$

Kecekungan



Kecekungan

- Teorema:

  Misalkan $z = f(x, y)$:

  - $d^2 z < 0 \Leftrightarrow f_{xx} < 0, f_{yy} < 0, f_{xx} f_{yy} > f_{xy}^2.$
  - $d^2 z > 0 \Leftrightarrow f_{xx} > 0, f_{yy} > 0, f_{xx} f_{yy} > f_{xy}^2.$

Ekstrem Lokal

- Fungsi 2-variabel: $z = f(x,y)$

(x^*, y^*) maksimum lokal jika:

Syarat orde-1 : $dz = 0$

$$f_x(x^*, y^*) = f_y(x^*, y^*) = 0$$

Syarat orde-2 : $d^2 z < 0$

$$f_{xx}(x^*, y^*) < 0, f_{yy}(x^*, y^*) < 0,$$

$$f_{xx}(x^*, y^*) f_{yy}(x^*, y^*) > f_{xy}^2(x^*, y^*)$$

(x^*, y^*) minimum lokal jika:

Syarat orde-1 : $dz = 0$

$$f_x(x^*, y^*) = f_y(x^*, y^*) = 0$$

Syarat orde-2 : $d^2 z > 0$

$$f_{xx}(x^*, y^*) > 0, f_{yy}(x^*, y^*) > 0,$$

$$f_{xx}(x^*, y^*) f_{yy}(x^*, y^*) > f_{xy}^2(x^*, y^*)$$

Ekstrem Lokal

- Tentukan ekstrem lokal dari $f(x,y) = x + 2ey - e^x - e^{2y}$.
- Diperoleh: $f_x = 1 - e^x, \quad f_y = 2e - 2e^{2y},$

  $f_{xx} = -e^x, \quad f_{yy} = -4e^{2y},$

  $f_{xy} = f_{yx} = 0.$
- Titik stasioner: $f_x = f_y = 0 \Leftrightarrow (x^*, y^*) = (0, \frac{1}{2}).$
- Syarat orde-2:

  $f_{xx}(0, \frac{1}{2}) = -1 < 0,$

  $f_{yy}(0, \frac{1}{2}) = -4e < 0,$

  $f_{xy}(0, \frac{1}{2}) = 0,$

  sehingga $f_{xx} f_{yy} > 0.$

(0,1/2) maksimum lokal.

Ekstrem Lokal

- Fungsi 3-variabel: $z = f(x_1, x_2, x_3)$
- Matriks gradien atau matriks Hess:

$$H = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}.$$

- Minor utama (leading principal minors):

$$|H_1| = |f_{11}| = f_{11}, |H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}, |H_3| = |H|.$$

Ekstrem Lokal

Maksimum Lokal

Di titik (x_1^*, x_2^*, x_3^*) berlaku:

- $f_1 = f_2 = f_3 = 0.$
- $|H_1| < 0, |H_2| > 0, |H_3| < 0.$

Minimum Lokal

Di titik (x_1^*, x_2^*, x_3^*) berlaku:

- $f_1 = f_2 = f_3 = 0.$
- $|H_1| > 0, |H_2| > 0, |H_3| > 0.$

Fungsi n-variabel: $z = f(x_1, x_2, \ldots, x_n)$

Condition	Maximum	Minimum
First-order necessary condition	$f_1 = f_2 = \cdots = f_n = 0$	$f_1 = f_2 = \cdots = f_n = 0$
Second-order sufficient condition[†]	$\lvert H_1\rvert < 0;\ \lvert H_2\rvert > 0;$ $\lvert H_3\rvert < 0;\ \ldots;\ (-1)^n\lvert H_n\rvert > 0$	$\lvert H_1\rvert, \lvert H_2\rvert, \ldots, \lvert H_n\rvert > 0$